

# SYZYGIES AND EQUATIONS OF KUMMER VARIETIES

SOFIA TIRABASSI

ABSTRACT. We study property  $N_p$  for Kummer varieties embedded by a power of an ample line bundle.

## 1. INTRODUCTION

Let  $X$  be an abelian variety. Its *associated Kummer variety*,  $K_X$ , is the quotient of  $X$  by the natural  $(\mathbb{Z}/2\mathbb{Z})$ -action induced by the morphism  $i_X : X \rightarrow X$  defined by  $x \mapsto -x$ . Given a Kummer variety,  $K_X$ , and an ample line bundle  $A$  on  $K_X$ , a result of Sasaki ([25]) states that  $A^{\otimes m}$  is very ample and the embedding it defines is projectively normal as soon as  $m \geq 2$ . Later Khaled ([15]) and Kempf ([14]) proved that, under the same conditions, the homogeneous ideal of  $K_X$  is generated by elements of degree 2 and 3, while, if  $m \geq 3$  it is generated in degree 2. Concerning the case  $m = 1$ , if, furthermore, we assume that  $A$  is a normally generated, very ample, line bundle on  $X$ , then the homogeneous ideal of  $K_X$  is generated in degree less than or equal to 4. In this paper we prove that these statements are particular cases of more general results on the syzygies of the variety  $K_X$ .

More precisely, let  $Z$  an algebraic variety over an algebraically closed field  $k$  and let  $\mathcal{A}$  an ample invertible sheaf on  $Z$ , generated by its global sections. With  $R_{\mathcal{A}}$  we will indicate the *section ring associated to the sheaf  $\mathcal{A}$* :

$$R_{\mathcal{A}} := \bigoplus_{n \in \mathbb{Z}} H^0(Z, \mathcal{A}^{\otimes n}),$$

while  $S_{\mathcal{A}}$  will be the symmetric algebra of  $H^0(Z, \mathcal{A})$ . The ring  $R_{\mathcal{A}}$  is a finitely generated graded  $S_{\mathcal{A}}$ -algebra and as such it admits a *minimal free resolution*

$$(1.1) \quad E_{\bullet} = 0 \rightarrow \dots \xrightarrow{f_{p+1}} E_p \xrightarrow{f_p} \dots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} R_{\mathcal{A}} \rightarrow 0$$

with  $E_i \simeq \bigoplus_j S_{\mathcal{A}}(-a_{ij})$ ,  $a_{ij} \in \mathbb{Z}_{>0}$ . In order to extend classical results of Castelnuovo, Mattuck, Fujita and Saint-Donat on the projective embeddings of curves, Green ([8]) introduced the following:

**Definition** (Property  $N_p$ ). Let  $p$  be a given integer. The line bundle  $\mathcal{A}$  satisfies property  $N_p$  if, in the notations above,

$$E_0 = S_{\mathcal{A}}$$

and

$$E_i = \bigoplus S_{\mathcal{A}}(-i-1) \quad 1 \leq i \leq p.$$

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Pareschi ([22]) extended the above condition as follows : we say that, given a non-negative integer  $r$ , property  $N_0^r$  holds for  $\mathcal{A}$  if, in the notation above,  $a_{0j} \leq 1 + r$  for every  $j$ . Inductively we say that  $\mathcal{A}$  satisfies property  $N_p^r$  if  $N_{p-1}^r$  holds for  $\mathcal{A}$  and  $a_{pj} \leq p + 1 + r$  for every  $j$ .

Green in [8] proved that, if  $Z$  is a smooth curve of genus  $g$  and  $\mathcal{A}$  a very ample line bundle on  $Z$  then  $\mathcal{A}$  satisfies  $N_p$  if  $\deg \mathcal{A} \geq 2g + 1 + p$ . He also conjectured the behaviour of property  $N_p$  for the canonical bundle of a non-hyperelliptic curve. Green's conjecture was recently proved for the general curve by Voisin ([27, 28]).

For what concerns higher dimensional varieties, the syzygies of the projective space were studied by Green in [9], [21], and in the recent preprint [5]. For arbitrary smooth varieties there is a general conjecture of Mukai and in [4] Ein–Lazarsfeld proved that, if  $Z$  is of dimension  $n$ , denoting by  $\omega_Z$  its canonical line bundle on  $Z$ , then for any  $\mathcal{L}$  very ample on  $Z$  the sheaf

$$\mathcal{A} := \omega_Z \otimes \mathcal{L}^{\otimes(n+1+d)}$$

satisfies  $N_p$  for every  $d \geq p \geq 1$ .

Abelian varieties distinguish themselves among other smooth varieties since, at least in what concerns their syzygies, they tend to behave in any dimension like elliptic curves. More precisely, Koizumi ([17]) proved that, given  $\mathcal{A}$  ample on an abelian variety  $X$ , then for  $m \geq 3$ ,  $\mathcal{A}^{\otimes m}$  embeds  $X$  in the projective space as a projectively normal variety. Furthermore, a classical theorem of Mumford ([20]), improved by Kempf ([11]), states that the homogeneous ideal of  $X$  is generated in degree 2 as long as  $m \geq 4$ . These results inspired Lazarsfeld to conjecture that  $\mathcal{A}^{\otimes m}$  satisfies  $N_p$  for every  $m \geq p + 3$ . A generalized version of Lazarsfeld's conjecture, involving property  $N_p^r$  rather than simply  $N_p$ , was proved in [22]; later in [24] Pareschi–Popa were able to recover and improve Pareschi's statements as a consequence of the powerful, Fourier-Mukai based, theory of  $M$ -regularity that they developed in [23].

Given the results on projective normality and degree of defining equations of Sasaki and Khaled, it was natural to conjecture that a bound  $m_0(p, r)$ , independent of the dimension of  $K_X$ , could be found such that  $\mathcal{A}^{\otimes m}$  satisfies  $N_p^r$  for every  $m \geq m_0(p, r)$ . In this paper we present some results in this direction. The main idea behind the proofs is that ample line bundles on a Kummer variety  $K_X$  have a nice description in terms of ample line bundles on  $X$ . More precisely, denoting by  $\pi_X : X \rightarrow K_X$  the quotient map, then for every  $A$  ample on  $K_X$  there exists  $\mathcal{A}$  ample on  $X$  such that  $\pi_X^* A \simeq \mathcal{A}^{\otimes 2}$ . Hence we can use Pareschi–Popa machinery to find some results on  $\mathcal{A}^{\otimes 2m}$  and then study how the  $\mathbb{Z}/2\mathbb{Z}$  action fits in the framework. Below we list the main results we obtained.

**Theorem A.** *Fix two non-negative integers  $p$  and  $r$  such that  $\text{char}(k)$  does not divide  $p + 1$ ,  $p + 2$ . Let  $A$  be an ample line bundle on a Kummer variety  $K_X$ . Then:*

- (a)  $A^{\otimes n}$  satisfies property  $N_p$  for every  $n \in \mathbb{Z}$  such that  $n \geq p + 2$ ;
- (b) more generally  $A^{\otimes n}$  satisfies property  $N_p^r$  for every  $n$  such that  $(r + 1)n \geq p + 2$ .

Since it consists in an improvement of existing results on the degrees of defining equations of Kummer varieties it is worth to emphasize individually the case  $p = 1$  of the above statement. Thanks to the geometric meaning of property  $N_p^r$  (Section 2.1) one can deduce the following:

**Particular Case B.** *Let be  $A$  a very ample line bundle on a Kummer variety  $K_X$ . Then the ideal of the image  $\varphi_A(K_X)$  in  $\mathbb{P}(H^0(X, A))$  is generated by forms of degree at most 4.*

The improvement with respect to what was classically known, thanks to the work of Wirtinger, Andreotti–Mayer [1] and Khaled, is that this holds for all Kummer varieties, and not only for the general ones. It might be interesting to see whether this fact has some applications to moduli of abelian varieties.

Adding one hypothesis about the line bundle  $A$  we can get a somewhat better result improving the work of Kempf and Khaled; namely:

**Theorem C.** *Let  $p$  and  $r$  be two integers such that  $p \geq 1$ ,  $r \geq 0$  and  $\text{char}(k)$  does not divide  $p+1$ ,  $p+2$ . Let  $A$  be an ample line bundle on a Kummer variety  $K_X$ , such that its pullback  $\pi_X^* A \simeq \mathcal{A}^{\otimes 2}$  with  $\mathcal{A}$  an ample symmetric invertible sheaf on  $X$  which does not have a base divisor. Then the followings hold:*

- (a)  $A^{\otimes n}$  satisfies property  $N_p$  for every  $n \in \mathbb{Z}$  such that  $n \geq p+1$ ;
- (b) more generally  $A^{\otimes n}$  satisfies property  $N_p^r$  for every  $n$  such that  $(r+1)n \geq p+1$ .

Again, it is worth of single out the case  $p = 1$  of the above Theorem, concerning the equations of the Kummer variety  $K_X$ .

**Particular Case D.** *Suppose that  $\text{char}(k)$  does not divide 2 or 3 and let  $A$  be an ample invertible sheaf on  $K_X$  such that  $\pi_X^* A \simeq \mathcal{A}^{\otimes 2}$  with  $\mathcal{A}$  without a base divisor. Then:*

- (a) if  $n \geq 2$ , then the ideal  $\mathcal{I}_{K_X, A^{\otimes n}}$  of the embedding  $\varphi_{A^{\otimes n}}$  is generated by quadrics;
- (b)  $\mathcal{I}_{K_X, A}$  is generated by quadrics and cubics.

The key point of the proofs of Theorems 2.A and 2.C will be to reduce the problem on the Kummer variety  $K_X$  to a different problem on the abelian variety  $X$ . Namely we will show that property  $N_p^r$  on the Kummer is implied by the surjectivity of a map of the type:

$$(*) \quad \bigoplus_{[\alpha] \in \widehat{U}} H^0(X, \mathcal{F} \otimes \alpha) \otimes H^0(X, \mathcal{H} \otimes \alpha) \xrightarrow{m_\alpha} H^0(X, \mathcal{F} \otimes \mathcal{H} \otimes \alpha)$$

where  $\mathcal{F}$  and  $\mathcal{H}$  are sheaves on  $X$  and  $\widehat{U}$  is a non empty open subset of  $\widehat{X}$ , the abelian variety dual to  $X$ , and  $m_\alpha$  is just the multiplication of global sections. Criteria for the surjectivity of such maps are implicit in Kempf's work ([12, 13]), for the case  $\mathcal{F}$  a vector bundle and  $\mathcal{H}$  a line bundle (for an explicit argument due to Lazarsfeld see [22]). These results were improved and extended to general coherent sheaves by Pareschi–Popa in [24].

This paper is organized in the following manner: in the next Section we explain some background material such as the relationship between property  $N_p^r$  and the cohomology of the Koszul complex and a useful criterion for the surjectivity of a map of type (\*). In Section 3 we present some slightly modified version of results of Sasaki and Khaled. The last section is entirely devoted to the proof of the main theorems.

*Notations.* Throughout the paper we will work over an algebraically closed field  $k$  of characteristic different from 2; further restrictions on the field will be stated when needed. The word "variety" will mean a projective variety over  $k$ .

- ◊ Given  $\mathcal{F}$  a sheaf on a variety  $Z$ , then we denote by  $\text{Bs}(\mathcal{F})$  the locus in  $Z$  where  $\mathcal{F}$  is not generated by its global sections, i. e. the locus of  $z \in Z$  where the evaluation map  $H^0(Z, \mathcal{F}) \otimes k(z) \rightarrow \mathcal{F} \otimes k(z)$  fails to be surjective.
- ◊ If  $L$  is a line bundle on a variety  $Y$ ,  $\varphi_L$  will be the rational map associated to  $L$ . The symbol  $\mathcal{I}_{Y, L}$  is the ideal of  $\varphi_L(Y)$  in  $\mathbb{P}(H^0(Y, L))$ .

- ◊ If we have a product of varieties,  $X_1 \times \cdots \times X_n$  then  $p_i$  is the  $i$ -th projection. Given  $\mathcal{F}_i$  a sheaf on  $X_i$ , then  $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n$  will stand for the sheaf  $p_1^* \mathcal{F}_1 \otimes \cdots \otimes p_n^* \mathcal{F}_n$  on  $X_1 \times \cdots \times X_n$ .

Given an abelian variety  $X$  of dimension  $g$ :

- ◊  $i_X$  will be the inversion morphism;  $t_x$  will be the translation by the point  $x$ ;
- ◊ The group law on the abelian variety is denoted by  $p_1 + p_2$ , while given an integer  $n$  the map  $n_X : X \rightarrow X$  will be the multiplication by  $n$ .

## 2. BACKGROUND MATERIAL

**2.1. Property  $N_p^r$  and Koszul Cohomology.** We begin by reviewing some well known relations between property  $N_p$ , or more generally property  $N_p^r$ , and the surjectivity of certain multiplication maps of sections of vector bundles. Let  $Z$  be a projective variety and  $L$  be an ample invertible sheaf on  $Z$ . We know from [8, Thm. 1.2] (see also [10] Thm. 1.2 or [18, p. 511]) that condition  $N_p$  is equivalent to the exactness in the middle of the complex

$$(2.1) \quad \bigwedge^{p+1} H^0(L) \otimes H^0(L^{\otimes h}) \rightarrow \bigwedge^p H^0(L) \otimes H^0(L^{\otimes h+1}) \rightarrow \bigwedge^{p-1} H^0(L) \otimes H^0(L^{\otimes h+2})$$

for any  $h \geq 1$ . More generally, condition  $N_p^r$  is equivalent to exactness in the middle of (2.1) for every  $h \geq r+1$ . Suppose that  $L$  generated by its global sections and consider the following exact sequence:

$$(2.2) \quad 0 \rightarrow M_L \rightarrow H^0(L) \otimes \mathcal{O}_Z \rightarrow L \rightarrow 0.$$

It is well known (see, for example [18] or [5]) that property  $N_p^r$  is implied by the surjectivity of

$$(2.3) \quad \bigwedge^{p+1} H^0(L) \otimes H^0(L^{\otimes h}) \rightarrow H^0 \left( \bigwedge^p M_L \otimes L^{\otimes h+1} \right).$$

It follows that if

$$(2.4) \quad H^1(Z, \bigwedge^{p+1} M_L \otimes L^{\otimes h}) = 0$$

for any  $h \geq r+1$ , then condition  $N_p^r$  is satisfied. If  $\text{char}(k)$  does not divide  $p$ ,  $\bigwedge^p \mathcal{E}$  is a direct summands of  $\mathcal{E}^{\otimes p}$  for any vector bundle  $\mathcal{E}$ . This yields

**Lemma 2.1.** *Assume that  $\text{char}(k)$  does not divide  $p+1$ .*

- (a) *If  $H^1(Z, M_L^{\otimes p+1} \otimes L^{\otimes h}) = 0$  for any  $h \geq r+1$ , then  $L$  satisfies  $N_p^r$ .*
- (b) *Let  $W \subseteq H^0(Z, L)$  be a free sublinear system and denote by  $M_W$  the kernel of the evaluation map  $W \otimes \mathcal{O}_Z \rightarrow L$ . Assume that  $H^1(Z, M_W^{\otimes p} \otimes L^{\otimes h}) = 0$ . Then  $H^1(Z, M_W^{\otimes p+1} \otimes L^{\otimes h}) = 0$  if and only if the multiplication map*

$$W \otimes H^0(M_W^{\otimes p} \otimes L^{\otimes h}) \rightarrow H^0(M_W^{\otimes p} \otimes L^{\otimes h+1})$$

*is surjective.*

*Proof.* The proof of (a) is straightforward, while (b) follows from the following exact sequence:

$$0 \rightarrow M_W^{\otimes p+1} \otimes L^{\otimes h} \rightarrow W \otimes M_W^{\otimes p} \otimes L^{\otimes h} \rightarrow M_W^{\otimes p} \otimes L^{\otimes h+1} \rightarrow 0.$$

□

2.1.1. *Property  $\mathbf{N}_{\mathbf{p}}^r$  for Small  $\mathbf{p}$ 's*. By definition, if a variety  $Z$  is embedded in a projective space by a very ample line bundle  $L$  satisfying property  $N_0^r$ , then the variety  $Z$  is  $h$ -normal for every  $h \geq r$ . In particular property  $N_0$  is equivalent to projective normality.

Although this is a standard fact, it is perhaps worth spelling out the meaning of property  $N_1^r$ , thus providing a direct proof of Particular Case B and D.

**Proposition 2.2.** *If  $L$  is a very ample line bundle on an algebraic variety  $Z$  satisfying  $N_1^r$ , then the homogeneous ideal of  $Z$  is generated by homogeneous elements of degree at most  $r + 2$ .*

*Proof.* Denote by  $V$  the vector space  $H^0(Z, L)$  and let  $S^k V$  be the component of degree  $k$  of the symmetric algebra of  $V$ ,  $S$ . Consider furthermore the two  $S$ -modules

$$I = \bigoplus H^0(\mathbb{P}(V), \mathcal{I}_{Z,L}(k))$$

and  $R_L$ . Look at the following commutative diagram where the middle column is given by the Koszul complex.

$$\begin{array}{ccccccc} & & S^{k-1}V \otimes \wedge^2 V & \xrightarrow{(3)} & H^0(Z, M_L \otimes L^{\otimes k}) & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & I_k \otimes V & \longrightarrow & S^k V \otimes V & \longrightarrow & \{R_L\}_k \otimes V & \longrightarrow 0 \\ & \downarrow (1) & & \downarrow (2) & & \downarrow & \\ 0 \longrightarrow & I_{k+1} & \longrightarrow & S^{k+1} V & \longrightarrow & \{R_L\}_{k+1} & \longrightarrow 0 \end{array}$$

Our aim is to see that the map (1) is surjective for every  $k \geq r + 2$ . Suppose that  $L$  satisfies property  $N_1^r$ , then in particular property  $N_0^r$  holds for  $L$  and the second and third row are exact for every  $k \geq r + 1$ . Since (2) is surjective, by the Snake Lemma, for every  $k \geq r + 1$  the surjectivity of (1) is implied by the surjectivity of (3) for every  $k \geq r + 2$ . Now we can factor (3) in the following way:

$$\begin{array}{ccc} S^{k-1}V \otimes \wedge^2 V & \xrightarrow{(3)} & H^0(Z, M_L \otimes L^{\otimes k}) \\ & \searrow g & \nearrow f \\ & H^0(Z, L^{\otimes k-1}) \otimes \wedge^2 V & \end{array}$$

where  $g$  is the canonical mapping  $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(k-1)) \longrightarrow H^0(Z, L^{\otimes k-1})$  and  $f$  is the map in (2.3). For every  $k \geq r + 2$ ,  $g$  is surjective because  $L$  satisfies  $N_0^r$ , while the surjectivity of  $f$  is equivalent to property  $N_1^r$ ; hence (3) is surjective.  $\square$

## 2.2. M-regular Sheaves and Multiplication Maps.

2.3. **A review on  $M$ -regularity.** In what follows we briefly recall some notions about  $M$ -regular sheaves. We refer to [23, 24] for further details.

Given an abelian variety  $X$  and a coherent sheaf  $\mathcal{F}$  on  $X$ , we introduce the *cohomological support loci* of  $\mathcal{F}$ :

$$V^i(\mathcal{F}) := \{\alpha \mid h^i(\mathcal{F} \otimes \alpha) \neq 0\} \subseteq \widehat{X}$$

It is said that a sheaf  $\mathcal{F}$  satisfies *index theorem* with index  $j$  (or equivalently it is said to be I.T.( $j$ )) if for every  $i \neq j$  the loci  $V^i(\mathcal{F})$  are empty.

**Definition 1.** A sheaf  $\mathcal{F}$  over an abelian variety  $X$  is said to be *Mukai-regular* (or simply *M-regular*) if, for every  $i > 0$ ,  $\text{codim } V^i(\mathcal{F}) \geq i + 1$ .

2.3.1. *Application to multiplication maps.* *M-regular* sheaves and *M-regularity* theory, introduced by Pareschi-Popa in a long series of paper ranging from 2002 to 2009, are crucial to our purpose thank to their application in determining whether a map of the form

$$(2.5) \quad \bigoplus_{[\alpha] \in U} H^0(X, \mathcal{F} \otimes \alpha) \otimes H^0(X, \mathcal{H} \otimes \alpha^\vee) \xrightarrow{m_\alpha} H^0(X, \mathcal{F} \otimes \mathcal{H}),$$

with  $\mathcal{F}$  and  $\mathcal{H}$  sheaves on an abelian variety  $X$  and  $U \subseteq \widehat{X}$  an open set, is surjective. We will list below all the results of such kind that we will be using throughout the paper. The first one is an extension of a theorem that had already appeared in the work of Kempf, Mumford and Lazarsfeld.

**Theorem 2.3** ([23], Theorem 2.5). *Let  $\mathcal{F}$  and  $\mathcal{H}$  be sheaves on  $X$  such that  $\mathcal{F}$  is *M-regular* and  $\mathcal{H}$  is locally free satisfying I.T. with index 0. Then (2.5) is surjective for any non empty Zariski open set  $U \subseteq \widehat{X}$ .*

*In particular, if  $\mathcal{F}$  and  $\mathcal{H}$  are as above, then there exists  $N$  a positive integer such that for the general  $[\alpha_1], \dots, [\alpha_N] \in \widehat{X}$  the map*

$$\bigoplus_{k=1}^N H^0(X, \mathcal{F} \otimes \alpha_k) \otimes H^0(X, \mathcal{H} \otimes \alpha_k^\vee) \xrightarrow{m_{\alpha_k}} H^0(X, \mathcal{F} \otimes \mathcal{H})$$

*is surjective.*

We conclude this paragraph by presenting two results on multiplication maps of sections we will be needing afterward.

**Proposition 2.4.** *Let  $\mathcal{A}$  be an ample line bundle on an abelian variety  $X$ . The map*

$$(2.6) \quad m : H^0(X, \mathcal{A}^{\otimes 2}) \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha).$$

*is surjective for the general  $[\alpha] \in \widehat{X}$ . If furthermore  $\mathcal{A}$  does not have a base divisor, then the locus  $Z \subseteq \text{Pic}^0(X)$  in which it fails to be surjective has codimension at least 2.*

*Proof.* The first part of the statement is classical, for a reference see, for example [2, Proposition 7.2.2]. Pareschi-Popa provided another proof of this, together with a proof of the second statement of the Proposition, using their Fourier-Mukai based methods (see [24, Proposition 5.2, Proposition 5.6, and Theorem 5.8]).  $\square$

### 3. MULTIPLICATION MAPS ON ABELIAN VARIETIES

Let  $\mathcal{A}$  be an ample symmetric invertible sheaf on the abelian variety  $X$  and denote by  $\psi_{\mathcal{A}} : \mathcal{A} \rightarrow i_X^* \mathcal{A}$  its *normalized isomorphism* (see [19, p. 304]). The  $\mathbb{Z}/2\mathbb{Z}$  action on  $X$  induced by the involution  $i_X : X \rightarrow X$  induces through  $\psi_{\mathcal{A}}$  a lifting of the action on  $\mathcal{A}$ . The composition

$$H^0(X, \mathcal{A}) \xrightarrow{i_X^*} H^0(X, i_X^* \mathcal{A}) \xrightarrow{i_X^*(\psi_{\mathcal{A}})} H^0(X, \mathcal{A})$$

is denoted by  $[-1]_{\mathcal{A}}$ . We let

$$H^0(X, \mathcal{A})^{\pm} = \{s \in H^0(X, \mathcal{A}) \text{ such that } [-1]_{\mathcal{A}} s = \pm s\}$$

If  $\mathcal{A}$  is *totally symmetric* (see [19, p. 304], e.g. if  $\mathcal{A}$  is an even power of a symmetric line bundle), then there exists<sup>1</sup> a line bundle  $A$  on the Kummer variety  $K_X$  such that  $\pi_X^* A \simeq \mathcal{A}$  and one can identify  $H^0(K_X, A)$  with  $H^0(X, \mathcal{A})^+$ . Conversely if  $A$  is an ample line bundle on  $K_X$ , then there exists  $\mathcal{A}$ , ample and symmetric on  $X$ , such that  $\pi_X^* A \simeq \mathcal{A}^{\otimes 2}$ .

A well known result by Khaled states that

**Proposition 3.1** ([15]). *Suppose that  $k = 2n$  is an even positive integer. Thus  $\mathcal{A}^{\otimes k}$  is totally symmetric. Then*

$$m_\alpha^+ : H^0(X, \mathcal{A}^{\otimes k})^+ \otimes H^0(X, \mathcal{A}^{\otimes h} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes k+h} \otimes \alpha)$$

is surjective for  $n \geq 1$ , every  $h \geq 3$ , and for every  $\alpha \in \text{Pic}^0(X)$

The main goal of this section is to prove that the same is true for every  $h = 2$  and for general  $\alpha \in \widehat{X}$ . If furthermore we assume that  $\mathcal{A}$  does not have a base divisor, then we will show that the locus of  $[\alpha] \in \widehat{X}$  where

$$(3.1) \quad m_\alpha^+ : H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 2(n+1)} \otimes \alpha)$$

fails to be surjective has codimension at least 2. We will do this by the methods of Khaled ([15]).

To this end, consider the isogeny

$$\xi : X \times X \longrightarrow X \times X$$

given by  $\xi = (p_1 + p_2, p_1 - p_2)$

**Lemma 3.2.** *For any  $[\alpha] \in \widehat{X}$  we have an isomorphism*

$$\xi^*(p_1^*(\mathcal{A} \otimes \beta) \otimes p_2^*(\mathcal{A} \otimes \alpha)) \longrightarrow p_1^*(\mathcal{A}^{\otimes 2} \otimes \beta \otimes \alpha) \otimes p_2^*(\mathcal{A}^{\otimes 2} \otimes \beta \otimes \alpha^\vee)$$

*Proof.* Observe that

$$\begin{aligned} \xi^*(p_1^*(\mathcal{A} \otimes \beta) \otimes p_2^*(\mathcal{A} \otimes \alpha))|_{X \times \{y\}} &\simeq t_y^* \mathcal{A} \otimes t_{-y}^* \mathcal{A} \otimes t_y^* \alpha^\vee \otimes t_{-y}^* \alpha \simeq \\ &\simeq \mathcal{A}^{\otimes 2} \otimes \beta \otimes \alpha. \end{aligned}$$

Restricting to  $\{0\} \times X$  we get

$$\xi^*(p_1^*(\mathcal{A} \otimes \beta) \otimes p_2^*(\mathcal{A} \otimes \alpha))|_{\{0\} \times X} \simeq \mathcal{A} \otimes \beta \otimes i^* \mathcal{A} \otimes \alpha \simeq \mathcal{A}^{\otimes 2} \otimes \beta \otimes \alpha^\vee;$$

the statement follows from the See-Saw principle.  $\square$

Composing with the Künneth isomorphism we have a map

$$\xi^* : H^0(X, \mathcal{A} \otimes \beta) \otimes H^0(X, \mathcal{A} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 2} \otimes \beta \otimes \alpha) \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \beta \otimes \alpha^\vee).$$

Taking  $[\alpha] = [\beta^\vee]$  we want to characterize the image of  $H^0(X, \mathcal{A} \otimes \beta) \otimes H^0(X, \mathcal{A} \otimes \beta^\vee)$  in  $H^0(X, \mathcal{A}^{\otimes 2})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \beta^{\otimes 2})$  through  $\xi^*$ .

In order to achieve this goal, we consider  $\widehat{T}^{\mathcal{A} \otimes \beta}$ , the involution of  $H^0(X, \mathcal{A} \otimes \beta) \otimes H^0(X, \mathcal{A} \otimes \beta^\vee)$  defined by  $\widehat{T}^{\mathcal{A} \otimes \beta}(s \otimes t) = i^*(\psi_{\beta^\vee})i^*t \otimes i^*(\psi_\beta)i^*s$ .

Let us denote by

$$[H^0(X, \mathcal{A} \otimes \beta) \otimes H^0(X, \mathcal{A} \otimes \beta^\vee)]^\pm$$

the eigenspaces.

<sup>1</sup>This is a standard fact. A proof could be found in [26, Prop. 2.1.7]



**Proposition 3.3.** *For every  $[\alpha] \in \widehat{X}$  we have*

$$\xi^*[H^0(X, \mathcal{A} \otimes \beta) \otimes H^0(X, \mathcal{A} \otimes \beta^\vee)]^\pm \subseteq H^0(X, \mathcal{A}^{\otimes 2})^\pm \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \beta^{\otimes 2}).$$

The proof of this statement is just a slight modification of Khaled's proof of [16, Proposition 2.2] and therefore we omit it. The complete proof can also be found in [26, Prop. 2.2.3]

**Theorem 3.4.** *Let  $\mathcal{A}$  be an ample symmetric line bundle on  $X$ . Take  $[\alpha] \in \widehat{X}$ . Then the multiplication map*

$$m : H^0(X, \mathcal{A}^{\otimes 2}) \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha)$$

*is surjective if and only if the following multiplication map is surjective*

$$m^+ : H^0(X, \mathcal{A}^{\otimes 2})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha)$$

*Proof.* Obviously, if the map  $m^+$  is surjective, then also  $m$  is. Therefore we are left to prove that the surjectivity of  $m$  implies the surjectivity of  $m^+$ . As observed by Khaled,  $m$  is the composition of

$$\xi^* : H^0(X, \mathcal{A}^{\otimes 2}) \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \rightarrow H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha) \otimes H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha^\vee)$$

with  $\text{id} \otimes e_{\mathcal{A}^{\otimes 4} \otimes \alpha^\vee}$ , where  $e_{\mathcal{A}^{\otimes 4} \otimes \alpha^\vee}$  denotes the evaluation in 0 of the sections of  $\mathcal{A}^{\otimes 4} \otimes \alpha^\vee$ . In fact, if we denote by  $\Delta : X \rightarrow X \times X$  the diagonal immersion, then  $m$  is just  $\Delta^*$  composed with the Künneth isomorphism. Now we can write  $\Delta = \xi \circ f$  where  $f : X \rightarrow X \times X$  is the morphism defined by  $x \mapsto (x, 0_X)$ . Now observe that, modulo Künneth isomorphism,

$$f^* : H^0(X, \mathcal{A}^{\otimes 4}) \otimes H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha^\vee) \rightarrow H^0(X, \mathcal{A}^{\otimes 4})$$

is exactly  $\text{id} \otimes e_{\mathcal{A}^{\otimes 4} \otimes \alpha^\vee}$ .

Observe that  $\xi \circ \xi$  is the map  $(x, y) \mapsto (2_X(x), 2_X(y))$ . Hence, taking  $\beta \in \text{Pic}^0(X)$  such that  $\beta^{\otimes 2} \simeq \alpha$ , we can write

$$m \circ \xi^* : H^0(X, \mathcal{A} \otimes \beta) \otimes H^0(X, \mathcal{A} \otimes \beta^\vee) \rightarrow H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha)$$

as  $\text{id} \otimes e_{\mathcal{A}^{\otimes 4} \otimes \alpha^\vee} \circ (2_X^*(-) \otimes 2_X^*(-)) = 2_X^* \otimes e_{\mathcal{A}^{\otimes 2} \otimes \beta^\vee}$ . Thus we can consider the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{[\beta^{\otimes 2}] = [\alpha]} H^0(\mathcal{A} \otimes \beta) \otimes H^0(\mathcal{A} \otimes \beta^\vee) & \xrightarrow{\xi^*} & H^0(\mathcal{A}^{\otimes 2}) \otimes H^0(\mathcal{A}^{\otimes 2} \otimes \alpha) \\ & \searrow 2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee} & \swarrow m \\ & H^0(\mathcal{A}^{\otimes 4} \otimes \alpha) & \end{array}$$

where the upper arrow is an isomorphism by projection formula. It follows that the surjectivity of  $m$  is equivalent to the surjectivity of

$$\bigoplus_{[\beta^{\otimes 2}] = [\alpha]} 2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee} : \bigoplus_{[\beta^{\otimes 2}] = [\alpha]} H^0(\mathcal{A} \otimes \beta) \otimes H^0(\mathcal{A} \otimes \beta^\vee) \rightarrow H^0(\mathcal{A}^{\otimes 4} \otimes \alpha),$$

which, in turn, is equivalent the following condition:

$$(\dagger) \quad 0 \notin \text{Bs}(\mathcal{A} \otimes \beta^\vee) \text{ for every } \beta \in \widehat{X} \text{ such that } \beta^{\otimes 2} \simeq \alpha.$$

We claim that  $(\dagger)$  implies the surjectivity of  $m^+$ , the statement of the theorem will follow directly.



Thanks to Proposition 3.3 and the isomorphism

$$\bigoplus_{[\beta^{\otimes 2}] \simeq [\alpha]} H^0(\mathcal{A} \otimes \beta) \xrightarrow{2_X^*} H^0(\mathcal{A}^{\otimes 4} \otimes \alpha),$$

yielded by  $2_X^*$  and projection formula, it is enough to check that for every  $[\beta]$  satisfying  $[\beta^{\otimes 2}] = [\alpha]$

$$2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee} [H^0(\mathcal{A} \otimes \beta) \otimes H^0(\mathcal{A} \otimes \beta^\vee)]^+ = 2_X^*(H^0(\mathcal{A} \otimes \beta)).$$

Hence for each  $s \in H^0(\mathcal{A} \otimes \beta)$ , we need a section  $\sigma \in (H^0(\mathcal{A} \otimes \beta) \otimes H^0(\mathcal{A} \otimes \beta^\vee))^+$  such that  $2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee}(\sigma) = 2_X^*(s)$ . Therefore take  $s \in H^0(X, \mathcal{A} \otimes \beta)$  and denote by  $\lambda$  the constant  $e_{\mathcal{A} \otimes \beta^\vee}([i^* \psi_\beta \circ i^*](s))$ . If  $\lambda \neq 0$ , take

$$\sigma := \frac{1}{\lambda} \cdot (s \otimes [i^* \psi_\beta \circ i^*](s)).$$

If, otherwise,  $\lambda = 0$ , it follows from (†) that there exists  $t \in H^0(X, \mathcal{A} \otimes \beta)$  such that  $e_{\mathcal{A} \otimes \beta^\vee}([i^* \psi_\beta \circ i^*](t)) = 1$ . Then, take  $\sigma$  to be the section

$$(s + t) \otimes [i^* \psi_\beta \circ i^*](s + t) - t \otimes [i^* \psi_\beta \circ i^*](t) \in (H^0(\mathcal{A} \otimes \beta) \otimes H^0(\mathcal{A} \otimes \beta^\vee))^+.$$

Applying  $2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee}$  to  $\sigma$  we get

$$\begin{aligned} 2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee}(\sigma) &= 2_X^* \otimes e_{\mathcal{A} \otimes \beta^\vee}((s + t) \otimes [i^* \psi_\beta \circ i^*](s + t) - t \otimes [i^* \psi_\beta \circ i^*](t)) = \\ &= 2_X^*(s + t) \cdot 1 - 2_X^*(t) \cdot 1 = 2_X^*(s). \end{aligned}$$

□

**Corollary 3.5.** (1) *For every  $\mathcal{A}$  ample symmetric invertible sheaf on  $X$  the multiplication map*

$$(3.2) \quad m^+ : H^0(X, \mathcal{A}^{\otimes 2})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 4} \otimes \alpha)$$

*is surjective for the generic  $[\alpha] \in \text{Pic}^0(X)$ .*

(2) *If furthermore  $\mathcal{A}$  does not have a base divisor, then the locus where (3.2) is not surjective has codimension at least 2.*

*Proof.* The statement follows directly from the Theorem 3.4 and the corresponding statements (see Proposition 2.4) about the map  $m$ . □

Now we are ready to prove the result that is the main point of this section.

**Theorem 3.6.** *Let  $\mathcal{A}$  be an ample symmetric invertible sheaf on  $X$ , then*

(1) *there exist a non-empty open subset  $U \subseteq \text{Pic}^0(X)$  such that for every  $n \in \mathbb{Z}$  with  $n \geq 1$  and and every  $\alpha \in U$  the following map is surjective*

$$(3.3) \quad m_\alpha^+ : H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 2n+2} \otimes \alpha);$$

(2) *if, furthermore,  $\mathcal{A}$  does not have a base divisor, then the locus  $Z$  in  $\text{Pic}^0(X)$  where (3.3) fails to be surjective has codimension at least 2.*

*Proof.* We will proceed by induction on  $n$ , with base case given by Corollary 3.5.

Case  $n > 1$ . Observe the following commutative diagram:

$$\begin{array}{ccc}
H^0(X, \mathcal{A}^{\otimes 2})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2(n-1)})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) & \xrightarrow{\varphi_\alpha} & H^0(X, \mathcal{A}^{\otimes 2})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2(n-1)+2} \otimes \alpha) \\
\downarrow & & \downarrow \psi_\alpha \\
H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, \mathcal{A}^{\otimes 2} \otimes \alpha) & \xrightarrow{m_\alpha} & H^0(X, \mathcal{A}^{\otimes 2n+2} \otimes \alpha)
\end{array}$$

Let  $[\alpha] \in \text{Pic}^0(X)$  be is such that  $m_\alpha$  is not surjective. Then, necessarily the map  $\psi_\alpha \circ \varphi_\alpha$  is not surjective. Since  $2 + 2(n-1) \geq 3$ , by Proposition 3.1,  $\psi_\alpha$  is surjective for every  $[\alpha] \in \widehat{X}$ . Per force, then, the map  $\varphi_\alpha$  could not be surjective. It follows that the locus of point  $[\alpha]$  such that  $m_\alpha$  is not surjective is contained in the following set:

$$W := \{[\alpha] \in \text{Pic}^0(X) \mid \varphi_\alpha \text{ is not surjective}\}.$$

By inductive hypothesis  $W$  has codimension  $\geq 1$ , in the general case, or  $\geq 2$ , when  $\mathcal{A}$  has not a base divisor. Therefore the Theorem is proved.  $\square$

#### 4. EQUATIONS AND SYZYGIES OF KUMMER VARIETIES

Putting together the results of the previous paragraphs, in this last Section we will prove Theorem A and Theorem C. First of all, observe that the case  $p = 0$  of the Theorem A follows directly as a corollary of Khaled's work (see Proposition 3.1). Thus we may suppose  $p \geq 1$ .

Our strategy will be to use the part (b) of Lemma 2.1 and to reduce the problem to checking the surjectivity of

$$(4.1) \quad H^0(K_X, A^{\otimes n}) \otimes H^0(K_X, M_{A^{\otimes n}}^{\otimes p} \otimes A^{\otimes nh}) \longrightarrow H^0(K_X, M_{A^{\otimes n}}^{\otimes p} \otimes A^{\otimes n(h+1)})$$

for every  $h \geq r+1$ . Let  $\mathcal{A}$  be an ample symmetric line bundle on  $X$  such that  $\mathcal{A}^{\otimes 2} \simeq \pi^* A$ , we split the proof in two steps.

In first place, the surjectivity of (4.1) is implied by the surjectivity of

$$(4.2) \quad H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, \pi^*(M_{A^{\otimes n}}^{\otimes p} \otimes \mathcal{A}^{\otimes 2nh})) \longrightarrow H^0(X, \pi^*(M_{A^{\otimes n}}^{\otimes p} \otimes \mathcal{A}^{\otimes 2n(h+1)})).$$

Before going any further we need to introduce some notation.

*Notation 4.1.* Suppose that  $A$  is an ample line bundle on  $K_X$  and let  $\mathcal{A}$  be an invertible sheaf on  $X$  such that  $\pi_X^* A = \mathcal{A}^{\otimes 2}$ . Take  $n$  an integer such that  $A^{\otimes n}$  is globally generated and consider the following exact sequence of vector bundles:

$$0 \rightarrow M_{A^{\otimes n}} \rightarrow H^0(K_X, A^{\otimes n}) \otimes \mathcal{O}_{K_X} \rightarrow A^{\otimes n} \rightarrow 0.$$

By pulling back via the canonical surjection  $\pi_X$  we get:

$$0 \rightarrow \pi_X^*(M_{A^{\otimes n}}) \rightarrow H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes \mathcal{O}_X \rightarrow \mathcal{A}^{\otimes 2n} \rightarrow 0.$$

Hence, after defining,

$$W_n := H^0(X, \mathcal{A}^{\otimes 2n})^+$$

we have that  $\pi_X^* M_{A^{\otimes n}} \simeq M_{W_n}$ .

From now on given a sheaf  $\mathcal{F}$  on  $X$ , we will often write  $H^0(\mathcal{F})$  instead of  $H^0(X, \mathcal{F})$ .

We will split the proof in two steps.

**Step 1:** *Reduction to an  $M$ -regularity problem*

Consider the vanishing locus  $V^1(M_{W_n} \otimes \mathcal{A}^{\otimes 2})$ . We claim that it coincides with the locus of  $[\alpha] \in \widehat{X}$  such that the multiplication map

$$m_\alpha^+ : H^0(\mathcal{A}^{\otimes 2n})^+ \otimes H^0(\mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(\mathcal{A}^{\otimes 2(n+1)} \otimes \alpha)$$

is not surjective. In fact, from the short exact sequence

$$0 \rightarrow M_{W_n} \otimes \mathcal{A}^{\otimes 2} \otimes \alpha \rightarrow W_n \otimes \mathcal{A}^{\otimes 2} \otimes \alpha \rightarrow \mathcal{A}^{\otimes 2(n+1)} \otimes \alpha \rightarrow 0.$$

taking cohomology we deduce that the surjectivity of  $m_\alpha^*$  is equivalent to the vanishing of  $H^1(M_{W_n} \otimes \mathcal{A}^{\otimes 2} \otimes \alpha)$ . Thanks to this characterization of the locus on  $\text{Pic}^0(X)$  where  $m_\alpha^+$  fails to be surjective we were able to prove the following

**Lemma 4.1.** *Let  $\mathcal{A}$  and  $\mathcal{E}$  be an ample symmetric sheaf on an abelian variety  $X$  and a coherent sheaf on  $X$ , respectively. If  $\mathcal{E} \otimes \mathcal{A}^{\otimes -2}$  is  $M$ -regular, then the multiplication map*

$$(4.3) \quad H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{A}^{\otimes 2n} \otimes \mathcal{E})$$

*is surjective for every  $n \geq 1$ .*

Before proceeding with the proof we will state an immediate corollary of this Lemma, that reduces our problem to a  $M$ -regularity problem.

**Corollary 4.2.** *If  $M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2(nh-1)}$  is  $M$ -regular, then (4.2) is surjective.*

*Proof of the Lemma.* By Proposition 3.6(1) we know that  $V^1(M_{W_n} \otimes \mathcal{A}^{\otimes 2})$  a proper closed subset of  $\widehat{X}$ . Therefore there exists an open set  $\widehat{U}_0 \subseteq \text{Pic}^0(X)$  such that  $\widehat{U}_0 \cap V^1(M_{W_n} \otimes \mathcal{A}^{\otimes 2}) = \emptyset$ . Now observe the following commutative diagram.

$$\begin{array}{ccc} \bigoplus_{\alpha \in \widehat{U}_0} H^0(\mathcal{A}^{\otimes 2n})^+ \otimes H^0(\mathcal{A}^{\otimes 2} \otimes \alpha) \otimes H^0(\mathcal{E} \otimes \mathcal{A}^{\otimes -2} \otimes \alpha^\vee) & & H^0(\mathcal{A}^{\otimes 2n})^+ \otimes H^0(\mathcal{E}) \\ \downarrow f & \searrow & \downarrow g \\ \bigoplus_{\alpha \in \widehat{U}_0} H^0(\mathcal{A}^{\otimes 2n+2} \otimes \alpha) \otimes H^0(\mathcal{E} \otimes \mathcal{A}^{\otimes -2} \otimes \alpha^\vee) & \xrightarrow{h} & H^0(\mathcal{E} \otimes \mathcal{A}^{\otimes 2n}) \end{array}$$

The map  $f = \oplus m_\alpha^+$  is surjective by our choice of the set  $\widehat{U}_0$ , the map  $h$  is surjective by  $M$ -regularity hypothesis together with Theorem 2.3. Thus  $g$  is necessarily surjective.  $\square$

**Step 3: Conclusions.** By Corollary 4.2 we are reduced to prove that  $M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2(nh-1)}$  is  $M$ -regular for every  $h \geq r$ . This is proved in the next two statements.

**Proposition 4.3.** *Let  $p$  be a positive integer. Then  $M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes m}$  satisfies I.T. with index 0 (and hence it is  $M$ -regular) for every  $m \geq 2p + 1$*

Note that Theorem A follows at once from this Proposition taking  $m = 2nr - 2$ .

*Proof.* We will proceed by induction on  $p$ .

*Case  $p = 1$ .* Let us consider the following exact sequence:

$$(4.4) \quad 0 \rightarrow M_{W_n} \otimes \mathcal{A}^{\otimes m} \otimes \alpha \rightarrow W_n \otimes \mathcal{A}^{\otimes m} \otimes \alpha \rightarrow \mathcal{A}^{\otimes 2n+m} \otimes \alpha \rightarrow 0.$$

One can easily see that the vanishing of the higher cohomology of  $M_{W_n} \otimes \mathcal{A}^{\otimes m} \otimes \alpha$  is implied by:

- (i) the vanishing of the higher cohomology of  $\mathcal{A}^{\otimes m} \otimes \alpha$  and
- (ii) the surjectivity of the following multiplication map:

$$H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, \mathcal{A}^{\otimes m} \otimes \alpha) \longrightarrow H^0(X, \mathcal{A}^{\otimes 2n+m} \otimes \alpha).$$

Condition (i) holds for every  $\alpha$  as long as  $m \geq 1$ , while, thanks to Khaled result (see Proposition 3.1), we know condition (ii) is satisfied for every  $\alpha$  as long as  $m \geq 3$ .

*Case  $p > 1$ .* Suppose now that  $p > 1$  and take any  $\alpha \in \widehat{X}$ . By twisting (4.4) by  $M_{W_n}^{\otimes p-1}$  we can observe that the vanishing of higher cohomology of  $M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes m} \otimes \alpha$  is implied by

- (i) the vanishing of the higher cohomology of  $M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes m} \otimes \alpha$  and
- (ii) the surjectivity of the following multiplication map:

$$(4.5) \quad H^0(X, \mathcal{A}^{\otimes 2n})^+ \otimes H^0(X, M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes m} \otimes \alpha) \longrightarrow H^0(X, M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2n+m} \otimes \alpha).$$

By induction (i) holds as long as  $m \geq 2p - 1$ . Thanks to Lemma 4.2 and Lemma 2.3 we know that if  $M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes m-2} \otimes \alpha$  satisfies I.T. with index 0, then (4.5) is surjective. But we use induction again and we get that  $M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes m-2} \otimes \alpha$  is I.T. with index 0 whenever  $m - 2 \geq 2p - 1$ , that is whenever  $m \geq 2p + 1$  and hence the statement is proved.  $\square$

**Proposition 4.4.** *In the notations above, take  $p \geq 1$  an integer. If  $\mathcal{A}$  does not have a base divisor, then  $M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes m}$  is  $M$ -regular for every  $m \geq 2p$ .*

Again Theorem C, follows at once after taking  $m = 2(nr - 1)$ .

*Proof.* For  $m \geq 2p + 1$  the statement is a direct consequence of the Proposition above, hence we can limit ourselves to the case  $m = 2p$ . We will proceed by induction on  $p$ .

*Case  $p = 1$ :* We want to prove that  $\text{codim } V^i(M_{W_n} \otimes \mathcal{A}^{\otimes 2}) > i$  for every  $i \geq 1$ . From the vanishing of the higher cohomology of  $\mathcal{A}^{\otimes 2} \otimes \alpha$  for every  $\alpha \in \text{Pic}^0(X)$  we know that the loci

$$V^i(M_{W_n} \otimes \mathcal{A}^{\otimes 2}) = \emptyset \quad \text{for every } i \geq 2.$$

Recall that the locus  $V^1(M_{W_n} \otimes \mathcal{A}^{\otimes 2})$  is the locus of points  $\alpha \in \widehat{X}$  such that the multiplication

$$W_n \otimes H^0(\mathcal{A}^{\otimes 2} \otimes \alpha) \longrightarrow H^0(A^{\otimes 2n+2} \otimes \alpha)$$

is not surjective. We know from Proposition 3.6 that, if  $\mathcal{A}$  has not a base divisor, then this locus has at least codimension 2 and hence the statement is proved.

*Case  $p > 1$*  Take  $\alpha \in \widehat{X}$ . Consider the following exact sequence

$$(4.6) \quad 0 \rightarrow M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2p} \otimes \alpha \longrightarrow W_n \otimes M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p} \otimes \alpha \longrightarrow M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p+n} \otimes \alpha \rightarrow 0$$

From Proposition 4.3(a) we know that for every  $i \geq 1$  both  $H^i(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p} \otimes \alpha)$  and  $H^i(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p+n} \otimes \alpha)$  vanish. Thus the loci  $V^i(M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2p})$  are empty for every  $i \geq 2$ . It remains to show that that

$$\text{codim } V^1(M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2p}) \geq 2.$$

As before one may observe that this locus is exactly the locus in  $\widehat{X}$  where the following multiplication map fails to be surjective:

$$W_n \otimes H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p} \otimes \alpha) \longrightarrow H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p+2n} \otimes \alpha).$$

In fact, taking cohomology in (4.6) and observing that for every  $[\alpha] \in \widehat{X}$ ,  $h^1(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p} \otimes \alpha) = 0$ , due to Proposition 4.3, we have the conclusion following the same argument as in the case  $p = 0$ .

Now take  $[\alpha] \in V^1(M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2p})$ . By inductive hypothesis the sheaf  $M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2(p-1)}$  is  $M$ -regular. Corollary 2.3 implies that there exist a positive integer  $N$  and  $[\beta_1], \dots, [\beta_N] \in \widehat{X}$  such that the following is surjective.

$$\bigoplus_{k=1}^N H^0(\mathcal{A}^{\otimes 2n+2} \otimes \beta_k \otimes \alpha) \otimes H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2(p-1)} \otimes \beta_k^\vee) \xrightarrow{m_{\beta_k}} H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p+2n} \otimes \alpha)$$

Consider the commutative square

$$\begin{array}{ccc} \bigoplus_{k=1}^N H^0(\mathcal{A}^{\otimes 2n})^+ \otimes H^0(\mathcal{A}^{\otimes 2} \otimes \alpha \otimes \beta_k) \otimes H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p-2} \otimes \beta_k^\vee) & & H^0(\mathcal{A}^{\otimes 2n})^+ \otimes H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2p} \otimes \alpha) \\ \downarrow & \searrow & \downarrow \\ \bigoplus_{k=1}^N H^0(\mathcal{A}^{\otimes 2n+2} \otimes \alpha \otimes \beta_k) \otimes H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2(p-1)} \otimes \beta_k^\vee) & \searrow & H^0(M_{W_n}^{\otimes p-1} \otimes \mathcal{A}^{\otimes 2n+2p}) \end{array}$$

The right arrow is not surjective by our choice of  $\alpha$ . The bottom arrow is surjective, hence the left arrow could not be surjective. Therefore

$$\alpha \in \bigcup_{k=1}^N Z_k,$$

where  $Z_k$  stands for the locus of  $[\beta] \in \widehat{X}$  such that the multiplication map

$$(4.7) \quad H^0(\mathcal{A}^{\otimes 2n})^+ \otimes H^0(\mathcal{A}^{\otimes 2} \otimes \beta \otimes \beta_k) \rightarrow H^0(\mathcal{A}^{\otimes 2n+2} \otimes \beta \otimes \beta_k)$$

fails to be surjective. Thus one has that

$$V^1(M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2p}) \subseteq \bigcup_{k=1}^N Z_k.$$

By Theorem 3.6(2) the loci  $Z_k$  have codimension at least 2, therefore

$$\text{codim } V^1(M_{W_n}^{\otimes p} \otimes \mathcal{A}^{\otimes 2p}) \geq \text{codim } \bigcup_{k=1}^N Z_k \geq 2$$

and the statement is proved.  $\square$

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INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, UL. BANACHA 2, 02-097 WARSZAWA (POLAND)  
*E-mail address:* `tirabassi@mimuw.edu.pl`